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Overstable hydromagnetic convection in a rotating fluid layer

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The effect of the simultaneous action of a uniform magnetic field and a uniform angular velocity on the linear stability of the Bénard layer to time-dependent convective motions is examined in the Boussinesq approximation. Four models, characterized by the relative directions of the magnetic field, angular velocity and gravitational force, are discussed under a variety of boundary conditions. Apart from a few cases, the treatment applies when the Taylor number T and the Chandrasekhar number Q (the square of the Hartmann number) are large. (These parameters are dimensionless measures of angular velocity and magnetic field, respectively.)

It is shown that the motions at the onset of instability can be of three types. If the Coriolis forces dominate the Lorentz forces, the results for the rotating non-magnetic case are retained to leading order. If the Coriolis and Lorentz forces are comparable, the minimum temperature gradient required for instability is *greatly reduced*. Also, in this case, the motions that ensue at marginal stability are necessarily three-dimensional and the Taylor–Proudman theorem and its analogue in hydromagnetics are no longer valid. When the Lorentz forces dominate the Coriolis forces, the results obtained are similar to those for the magnetic non-rotating case at leading order.

The most unstable mode is identified for all relations $T = KQ^{\alpha}$, where K and α are positive constants, taking into account both time-dependent and time-independent motions.

Various types of boundary layers developing on different boundaries are also examined.

1. Introduction

'Bénard layer' is a term used to describe a layer of viscous fluid, of thickness d, contained between two horizontal planes when an adverse temperature gradient β is applied across it. Since Bénard's famous experiments in 1900, considerable work has been reported on the Bénard layer. In 1916, Rayleigh gave the first mathematical formulation of the linear stability of the Bénard layer to convective motions of infinitesimal amplitude, using the Boussinesq approximation. At the onset of instability, these convective motions can be of two types. If they are independent of time, it is usually said that the principle of exchange of stabilities is valid. If, however, instability manifests itself in the form of oscillatory (with respect to time) motions, we have overstability. A review of all the work done on the Bénard layer, including the effects of a uniform magnetic field $\mathbf{B}_0$ or a uniform angular velocity $\mathbf{\Omega}$ antiparallel to the gravitational force, can be found in Chandrasekhar (1961) and in Weiss (1964).

The simultaneous action of rotation and a uniform magnetic field on the Bénard layer was first studied by Chandrasekhar (1954), for the case when the bounding planes are free. The parameters characterizing the flow are the Prandtl number p, the magnetic Prandtl number p_m, the Chandrasekhar number Q, the Taylor number T and the Rayleigh number R. They are defined by

$$p = \nu/\kappa, \quad p_m = \nu/\eta, \quad Q = d^2B^2/\mu\rho\nu\eta = M^2, \quad T = 4d^2\Omega^2/\nu^2, \quad R = g\bar{\alpha}\beta d^4/\nu\kappa, \tag{1.1}$$

where ν is the kinematic viscosity, κ the thermal diffusivity, μ the magnetic permeability, ρ the mean density, η ($= 1/\mu\sigma_e$, σ_e being the electric conductivity) the magnetic diffusivity, g the gravitational acceleration, $\bar{\alpha}$ the coefficient of thermal expansion and B and Ω are the magnitudes of $\mathbf{B}_0$ and $\mathbf{\Omega}$, respectively. In (1.1), M is the usual Hartmann number. The problem of the magnetic rotating Bénard layer is one of the few cases of interaction of a field, rotation and adverse temperature gradient amenable to exact treatment. Furthermore, this interaction is relevant to theories of convective motions in the earth's core and planetary interiors where these three ingredients are known to exist. In turn, convection may drive the geomagnetic dynamo which maintains the earth's magnetic field (Soward 1974).

In contrast to the inhibiting effects on the layer of a uniform magnetic field or uniform rotation acting separately, the simultaneous action of these fields *facilitates* convection (Eltayeb & Roberts 1970). Eltayeb (1972*a*, hereafter referred to as I) examined the linear stability of the hydromagnetic rotating layer when the principle of exchange of stabilities is valid, for different types of boundaries. In I, four different models, characterized by the directions of the angular velocity and magnetic field relative to that of gravity, were discussed. For each model, the critical mode was located for all relations $T = T(Q)$ in the double limit $T, Q \to \infty$.

The four models studied here, which are the same as those considered in I, are defined by

(I) $\mathbf{B}_0$ vertical, $\mathbf{\Omega}$ vertical,

(II) $\mathbf{B}_0$ horizontal, $\mathbf{\Omega}$ vertical,

(III) $\mathbf{B}_0$ horizontal, $\mathbf{\Omega}$ horizontal with an angle ϕ between them,

(IV) $\mathbf{B}_0$ vertical, $\mathbf{\Omega}$ horizontal.

We shall extend the discussion of I to include the possibility of overstability for non-zero values of the magnetic Prandtl number p_m. We should emphasize that the present work also applies in the double limit $Q, T \to \infty$, although models II and III include cases when exact solutions exist for all values of T and Q. In model IV, the marginal state has the form of a roll parallel to Ω and the results for the non-rotating magnetic layer apply (see Gibson 1966). We shall not pursue this model any further. In the other models, it is found that the infinity of relations $T = T(Q)$ may conveniently be divided into three distinct cases. When the Coriolis forces dominate the Lorentz forces, the critical mode, according to the linear stability theory, is similar to that obtained in the absence of the magnetic field, to leading order. In this case, the critical Rayleigh number R_c varies as the two-thirds power of the Taylor number and the tessellated cell pattern is elongated in the vertical direction. Another case arises when the Lorentz forces dominate the Coriolis forces, resulting in a critical mode identical to that obtained in the absence of rotation. Here R_c varies like Q and again the cell pattern prevalent at marginal stability is elongated in the vertical direction. The third case occurs when the Coriolis and Lorentz forces are comparable. In this situation, the critical Rayleigh number R_c varies as the larger of the quantities T/Q and $T^{\frac{1}{2}}$; the cell pattern obtained has dimensions of the order of the thickness of the layer.

For model I, in which the angular velocity and magnetic field are both vertical, the above results are obtained for *all* types of boundaries. For models II and III, however, we only consider the case of perfectly conducting boundaries.

The imposition of the condition $Q, T \to \infty$ naturally leads to the development of boundary layers. The nature of these boundary layers depends on the relation between T and Q, on the relative directions of the relevant forces and on the dimensions of the prevailing cell pattern. For example, in model I, we find that Hartmann layers develop on all types of boundary when the Lorentz forces are dominant, otherwise Ekman layers are present; in model II, Ekman layers are always present, while model III is associated with Stewartson layers. Other diffusion layers of various types are also examined.

In §2, the basic equations and boundary conditions of the problem are set out; in §3, we examine model I; in §4, we discuss model II, while §5 is devoted to model III. In §6, we survey the similarities and differences between the different models considered.

2. The basic equations, boundary conditions and notation

Consider a Bénard layer of electrically conducting fluid rotating with uniform angular velocity $\mathbf{\Omega}$. Take a Cartesian co-ordinate system rotating with the fluid layer with origin O half-way between the top and bottom planes, z axis vertically upwards and x and y axes in any two perpendicular horizontal directions. All quantities in this paper are measured in this rotating frame.

When the Bousinnesq approximation is applicable, the equations of the problem (see I) admit a steady solution in which heat is *conducted* uniformly from the lower plane to the upper plane in the presence of a uniform magnetic field $\mathbf{B}_0$. The equations governing small perturbations of this solution are given in I. If we express each variable X as a normal mode of the form

$$X(x,y,z,t) = X(z)\exp i(kx+ly+\sigma t), \tag{2.1}$$

the basic equations of the problem can be written in dimensionless form as

$$L\zeta = T^{\frac{1}{2}}(i\sigma p_m - \nabla^2)(\hat{\mathbf{\Omega}}.\nabla)W, \tag{2.2}$$

$$L\xi = T^{\frac{1}{2}}(\hat{\mathbf{B}}.\nabla)(\hat{\mathbf{\Omega}}.\nabla)W, \tag{2.3}$$

$$(i\sigma p_m - \nabla^2)\, b = (\hat{\mathbf{B}}\,.\,\nabla)\, W, \tag{2.4}$$

$$\{(i\sigma p - \nabla^2)\,\nabla^2 L^2 + T(i\sigma p_m - \nabla^2)^2\,(\hat{\boldsymbol{\Omega}}\,.\,\nabla)^2\}\, W = Ra^2(i\sigma p_m - \nabla^2)\, LW, \tag{2.5}$$

where
$$L = (i\sigma p_m - \nabla^2)\,(i\sigma - \nabla^2) - Q(\hat{\mathbf{B}}\,.\,\nabla)^2 \tag{2.6}$$

and
$$a^2 = k^2 + l^2, \quad D \equiv d/dz, \quad \nabla^2 \equiv D^2 - a^2. \tag{2.7}$$

Here $\hat{\mathbf{B}}$ and $\hat{\boldsymbol{\Omega}}$ are unit vectors in the directions of $\mathbf{B}_0$ and $\boldsymbol{\Omega}$, respectively. W, b, ξ and ζ are, respectively, the vertical components of the velocity $\mathbf{u}$, the magnetic field $\mathbf{b}$, the electric current $\mathbf{J}$ ($=$ curl $\mathbf{h}/\mu$) and the vorticity ($=$ curl $\mathbf{u}$); θ is the perturbation in temperature. p, p_m, R, T and Q have already been defined in the introduction. From now onwards, we shall assume that all the variables are functions of z only and suppress the exponential dependence.

Equations (2.2)–(2.7) must be solved subject to certain boundary conditions. We shall always assume that the lower and upper boundaries (now located at $z = \pm\frac{1}{2}$) are of the same nature. As was shown in I, the thermal and dynamic boundary conditions are

$$\theta = W = D^2 W = D\zeta = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{2.8}$$

if the boundaries are free (no tangential stress), and

$$\theta = W = DW = \zeta = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{2.9}$$

when the boundaries are rigid.

The magnetic boundary conditions (cf. Roberts 1967, chap. 1) are, in general, the continuity of $\mathbf{n} \wedge \mathbf{E}$, $\mathbf{n}\,.\,\mathbf{b}$ and $\mathbf{n} \wedge \mathbf{b}/\mu$, where $\mathbf{E}$ is the electric field and $\mathbf{n}$ is a unit vector normal to the boundary. Although these constitute six conditions, it can be shown that they are equivalent to only two boundary conditions (see Eltayeb 1972*b*, chap. 2). Furthermore, the boundary conditions obtained by Eltayeb can be simplified further by using equation (2.4) above† to get

$$Db \pm \gamma b = \bar{p}(D\xi + h_1\zeta) \pm p_m \gamma \xi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}, \tag{2.10}$$

where
$$\gamma^2 = a^2 + i\sigma\bar{p}, \quad \bar{p} = \nu\mu\sigma_e'. \tag{2.11}$$

Here σ_e' is the electrical conductivity of the boundary and h_1 is the cosine of the angle $\mathbf{B}_0$ makes with the z axis. We have assumed that the magnetic permeability of the boundary is the same as that of the layer. It is worth noting that the first condition in (2.10) is the same as that used by Gibson (1966). The second condition is automatically satisfied by Gibson's solutions since ξ and ζ vanish identically in the absence of rotation. In the treatment by Chandrasekhar the second condition is satisfied in the special cases considered while the first is violated for solutions in which $p_m \neq 0$.

Because of the complexity of the problem, we shall find it useful to introduce a notation to be adopted in all the models. The boundary conditions examined here are of one of the following types.

(*A*) Free boundaries with finite electrical conductivity, for which

$$W = \theta = D^2 W = D\zeta = \Lambda_1 = \Lambda_2 = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.12}$$

† The exact details of the derivation of (2.10) are available from the author or the JFM Editorial Office, DAMTP, Silver Street, Cambridge CB3 9EW.

(*B*) Free, electrically insulating boundaries, for which

$$W = \theta = D^2W = D\zeta = Db \pm ab = \xi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.13}$$

(*C*) Free, perfectly conducting boundaries, for which

$$W = \theta = D^2W = D\zeta = b = \Lambda_3 = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.14}$$

(*D*) Rigid boundaries with finite electrical conductivity, for which

$$W = \theta = DW = \zeta = \Lambda_1 = \Lambda_4 = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.15}$$

(*E*) Rigid insulating boundaries, for which

$$W = \theta = DW = \zeta = Db \pm ab = \xi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.16}$$

(*F*) Rigid, perfectly conducting boundaries, for which

$$W = \theta = DW = D\zeta = b = D\xi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.17}$$

Here Λ_1 and Λ_2 denote the expressions in (2.11), respectively, $\Lambda_3 = D\xi \pm h_1\zeta$ and Λ_4 is obtained from Λ_2 by putting $\zeta = 0$.

Equations (2.2)–(2.7) together with the appropriate boundary conditions pose a double eigenvalue problem for R and σ. For each pair of wavenumbers k and l, these equations possess a non-trivial solution satisfying the relevant boundary conditions if R and σ take certain values which depend on T, Q, p and p_m. Since the method of locating the minimum Rayleigh number was discussed in I, we shall restrict ourselves here to a few points. For fixed values of T and Q, and provided that p and p_m satisfy a specified relation, a discrete set of solutions W_n exists and each solution is characterized by a Rayleigh number $R^{(n)}(k, l, \sigma)$. We first find the solution with the smallest Rayleigh number $R^{(0)}(k, l, \sigma)$ and minimize the Rayleigh number for this solution over all possible values of k and l for which σ^2 is positive. The critical mode, according to the linear theory, is then defined by this minimum Rayleigh number R_c and the corresponding values k_c, l_c and σ_c of k, l and σ. To find the critical mode at every point of the T, Q plane where T and Q are large, it is found that if we assume the relation

$$T = KQ^{\alpha}, \tag{2.18}$$

where K and α are positive constants, then the range of α can conveniently be divided into three main regions. In general, these regions are where (i) magnetic effects dominate, (ii) magnetic and rotational effects are comparable and (iii) rotational effects dominate. We shall always use these small roman numbers when we refer to them. The notation is therefore similar to that in I except for the classification of the types of boundary under consideration. This disagreement in notation is due to the fact that boundaries of finite conductivity were not discussed in I.

A useful property of (2.2)–(2.7) and the associated boundary conditions, for the models considered here, is that the set of solutions falls into two uncoupled groups: even and odd solutions. This allows us to consider the half-interval $0 \leqslant z \leqslant \frac{1}{2}$. When steady convection was discussed in I it was found that although the even-mode solution gave the critical mode in most situations, the odd mode

Model	Case	Range of applicability	Critical Rayleigh number, R_c	(Critical frequency)^2 σ_c^2	Critical wavenumber a_c	Section
I	A, D(i)	$T \leqslant 62{\cdot}8(p_m^2 Q^5/\overline{p}^2)^{\frac{1}{4}}$	$\sim Q/q^2$	$\sim Q/p_m$	$\sim (p^8 Q^5/p\overline{p}^2)^{\frac{1}{24}}$†	3
	A, D(ii, a)	$62{\cdot}8(p_m^2 Q^5/\overline{p}^2)^{\frac{1}{4}} \leqslant T \leqslant 0{\cdot}007p(p^3/p_m\overline{p}^2)\, Q^2/p_m$	$\sim 2Q/q^2$	$\sim Q^2 a_c^2/p_m^2 T$	$\sim p_m/\overline{p}(qT/Q^2 - Q/p_m^2 T)^2$	3
	A, D(ii, b)	$0{\cdot}007p(p^3/p_m\overline{p}^2)^{\frac{1}{5}}\, Q^2/p_m \leqslant T \leqslant 1{\cdot}47p^4Q^3$	$\sim 17{\cdot}6T/Q$	$= 0$	$\sim 2{\cdot}328$	3
	A, D(iii)	$T \geqslant 1{\cdot}47p^4Q^3$	$\sim 3{\cdot}8(p^2T)^{\frac{2}{3}}$	$\sim 1{\cdot}26(T/p^2)^{\frac{2}{3}}$	$\sim 0{\cdot}89(p^2T)^{\frac{1}{6}}$	3
	B, E(i)	$T \leqslant 0{\cdot}07(p^2Q^{11}/p_m^4)^{\frac{1}{10}}/q$	$\sim Q/q^2$	$\sim Q/p_m$	$\sim (p^2Q/p_m)^{\frac{1}{5}}$†	3
	B, E(ii, a)	$0{\cdot}07(p^2Q^{11}/p_m^4)^{\frac{1}{10}}/q \leqslant T \leqslant (p^2Q^3/p_m^5)^{\frac{1}{2}}$	$\sim Q/q^2$	$\sim Q/p_m$	$\sim (p_m p^4 T^2 Q)^{\frac{1}{11}}$†	3
		$(p^2Q^3/p_m^5)^{\frac{1}{2}} \leqslant T \leqslant 4p^2Q^2/p_m^2$	$\sim 2Q/q^2$	$\sim Q^2a_c^2/p_m^2 T$	$\sim (Q^2/q^2T)^{\frac{1}{4}}$†	3
	B, E(ii, b)	$4p^2Q^2/p_m^2 \leqslant T \leqslant 0{\cdot}84p^4Q^3$	$\sim 4T/Q$	$= 0$	$= 1{\cdot}0$	3
	B, E(iii)	$T \geqslant 0{\cdot}84p^4Q^3$	$\sim 3{\cdot}8(p^2T)^{\frac{2}{3}}$	$\sim 1{\cdot}26(T/p^2)^{\frac{2}{3}}$	$\sim 0{\cdot}89(p^2T)^{\frac{1}{6}}$	3
	C, F(i)	$T \leqslant 0{\cdot}07(p^2Q^{11}/p_m^4)^{\frac{1}{10}}/q$	$\sim Q/q^2$	$\sim Q/p_m$	$\sim 1{\cdot}5(p^2Q/p_m)^{\frac{1}{5}}$†	3
	C, F(ii, a)	$0{\cdot}07(p^2Q^{11}/p_m^4)^{\frac{1}{10}}/q \leqslant T \leqslant (p^2Q^3/p_m^5)^{\frac{1}{2}}$	$\sim Q/q^2$	$\sim Q/p_m$	$\sim 1{\cdot}37(p_m p^4T^2Q)^{\frac{1}{11}}$†	3
		$(p^2Q^3/p_m^5)^{\frac{1}{2}} \leqslant T \leqslant 0{\cdot}11Q^2/q^2$	$\sim 2Q/q^2$	$\sim Q^2a_c^2/p_m^2 T$	$\sim (Q^2/q^2T)^{\frac{1}{4}}$†	3
	C, F(ii, b)	$0{\cdot}11Q^2/q^2 \leqslant T \leqslant 1{\cdot}47p^4Q^3$	$\sim 17{\cdot}6T/Q$	$= 0$	$\sim 2{\cdot}328$	3
	C, F(iii)	$T \geqslant 1{\cdot}47p^4Q^3$	$\sim 3{\cdot}8(p^2T)^{\frac{2}{3}}$	$\sim 1{\cdot}26(T/p^2)^{\frac{2}{3}}$	$\sim 0{\cdot}89(p^2T)^{\frac{1}{6}}$	3
II	C, F(i)	$T \leqslant 1{\cdot}33Q^2/q^2$	$\sim 10{\cdot}4T^{\frac{1}{2}}/q^2$	$\sim 9/q$	$a_c \sim 1{\cdot}4,\quad k_c \sim 1{\cdot}3(q^2T/Q^2)^{\frac{1}{4}}$	4
	C, F(ii)	$1{\cdot}33Q^2/q^2 \lesssim T \leqslant 54p^4Q^3$‡	$\sim T/Q$	$= 0$	$k_c \sim 1{\cdot}19(T/Q^2)^{\frac{1}{4}},\quad l_c = 0$	4
	C, F(iii)	$T \geqslant 54p^4Q^3$	$\sim 3{\cdot}8(p^2T)^{\frac{2}{3}}$	$\sim 1{\cdot}26(T/p^2)^{\frac{2}{3}}$	$k_c = 0,\quad l_c \sim 0{\cdot}89(p^2T)^{\frac{1}{6}}$	4
III§	C, F(i)	$T \leqslant 0{\cdot}062Q^2/q^2$	$\sim 2h(a_c)\, Q/q^2$‖	$\sim (1+a_c^2)$ $(1-a_c^4)/a_c^2$	$a_c \sim 0{\cdot}7(1-0{\cdot}43T^{\frac{1}{2}}q/Q)^{\frac{1}{2}}$ $k_c \sim -1{\cdot}3(q^2T/Q^2)^{\frac{1}{4}}$	5
	C, F(ii)	$T \geqslant 0{\cdot}062Q^2/q^2$	$\sim Q/q^2$	$\sim (Q/p_m)^{\frac{1}{2}}/p$	$k_c \sim (p_m/p^2Q)^{\frac{1}{4}},\quad l_c = 0$§	5

† These values are obtained by considering second-order terms in the expression for R.
‡ The lower boundary of this case is not precise.
§ The values entered here correspond to $\phi = 90°$.
‖ $h(a) = (1+a^2)(1-2a^2)/(1-a^2)^2$.

TABLE 1. Central results when $p \ll p_m \ll 1$. ($q = p_m/p$)

			Outer layer		Intermediate layer		Inner layer	
Model	Case	Quantities zero in mainstream at $z = \frac{1}{2}$	Thickness	Quantities brought to zero at $z = \frac{1}{2}$	Thickness	Quantities brought to zero at $z = \frac{1}{2}$	Thickness	Quantities brought to zero at $z = \frac{1}{2}$
I	A(i), (ii, a)	W, Λ_2	a^{-1}	θ	$\sigma^{-\frac{1}{2}}$	Λ_1	$Q^{-\frac{1}{2}}$	$D^2W, D\zeta$
	A(ii, b)	W, θ, Λ_2	—	None	—	None	$T^{-\frac{1}{4}}$	$D^2W, D\zeta$
	A(iii)	W	a^{-1}	$\theta, \Lambda_1, \Lambda_2$	—	None	$T^{-\frac{1}{4}}$	$D^2W, D\zeta$
	B(i), (ii, a)	W, ξ	a^{-1}	θ	$\sigma^{-\frac{1}{2}}$	$Db+ab$	$Q^{-\frac{1}{2}}$	$D^2W, D\zeta$
	B(ii, b)	W, θ	—	None	—	None	$T^{-\frac{1}{4}}$	$D^2W, D\zeta$
	B(iii)	W	a^{-1}	$\theta, \xi, Db+ab$	—	None	$T^{-\frac{1}{4}}$	$D^2W, D\zeta$
	C(i), (ii, a)	$W, \zeta+D\xi$	a^{-1}	θ	$\sigma^{-\frac{1}{2}}$	b	$Q^{-\frac{1}{2}}$	$D^2W, D\zeta$
	C(ii, b)	$W, \theta, \zeta+D\xi$	—	None	—	None	$T^{-\frac{1}{4}}$	$D^2W, D\zeta$
	C(iii)	W	a^{-1}	$b, \theta, D\xi+\zeta$	—	None	$T^{-\frac{1}{4}}$	$D^2W, D\zeta$
	D(i), (ii, a)	W, Λ_4	a^{-1}	θ	$\sigma^{-\frac{1}{2}}$	Λ_1	$Q^{-\frac{1}{2}}$	DW, ζ
	D(ii, b)	W, θ, Λ_4	—	None	—	None	$T^{-\frac{1}{4}}$	DW, ζ
	D(iii)	W	a^{-1}	$\Lambda_4, \Lambda_1, \theta$	—	None	$T^{-\frac{1}{4}}$	DW, ζ
	E(i), (ii, a)	W, ξ	a^{-1}	θ	$\sigma^{-\frac{1}{2}}$	$Db+ab$	$Q^{-\frac{1}{2}}$	DW, ζ
	E(ii, b)	W, θ, ξ	—	None	—	None	$T^{-\frac{1}{4}}$	DW, ζ
	E(iii)	W	a^{-1}	$Db+ab, \theta, \xi$	—	None	$T^{-\frac{1}{4}}$	DW, ζ
	F(i), (ii, a)	$W, D\xi$	a^{-1}	θ	$\sigma^{-\frac{1}{2}}$	b	$Q^{-\frac{1}{2}}$	DW, ζ
	F(ii, b)	$W, \theta, D\xi$	—	None	—	None	$T^{-\frac{1}{4}}$	DW, ζ
	F(iii)	W	a^{-1}	$b, \theta, D\xi$	—	None	$T^{-\frac{1}{4}}$	DW, ζ
II	C	$W, \theta, D^2W, D\zeta$	—	None	—	None	—	None
	F(i)	$b, D\xi$						
	F(i)	$W, \theta, b, D\xi$	—	None	—	None	$T^{-\frac{1}{4}}$	DW, ζ
	F(ii) } F(iii) }	W	a^{-1}	$\theta, (D\xi)$†	—	None	$T^{-\frac{1}{4}}$	$DW, \zeta, (D\xi)$†
III	C(i)	W, θ, b	—	None	—	None	$T^{-\frac{1}{6}}$	$D^2W, D\zeta, D\xi$
	C(ii)‡	W, θ, b, D^2W	—	None	—	None	—	None
		or						
		W	$\sigma^{-\frac{1}{2}}$	θ, b	—	None	$T^{-\frac{1}{6}}$	$D^2W, D\zeta, D\xi$
	F(i)	W, θ, b	—	None	—	None	$T^{-\frac{1}{6}}$	$DW, \zeta, D\xi$
	F(ii)‡	W	—	None	—	None	$Q^{-\frac{1}{8}}$	b, θ, DW
		or						
		W	$\sigma^{-\frac{1}{2}}$	θ, b	—	None	$T^{-\frac{1}{6}}$	$DW, \zeta, D\xi$

† Quantities in parentheses are brought to zero on the boundary by the two layers together.
‡ The first mode is preferred if $p_m < 1$, otherwise the second is preferred.

TABLE 2. Boundary conditions and boundary layers

was preferred in some cases. In the case of overstability, however, we find that the *even* mode is always the easiest to excite.

All six types of boundaries A–F are examined for model I. In the analysis below, however, we shall only deal with A, B and C in detail and indicate the differences with a rigid boundary, if any, at the end of the corresponding free-boundary treatment. The critical modes for free and rigid boundaries of the same conductivity are always the same as can be seen in table 1. In models II and III, only types C and F are examined. In the text, only type C is discussed but both C and F are included in table 2. The cases (iii) of rotational dominance in both models I and II are included in tables 1 and 2 for comparison purposes. The critical modes here are, to leading order, identical with those in the absence of the magnetic field (see Chandrasekhar 1961, chap. 3).

The eigenfunctions of all the models considered, in most of the cases, have a z dependence whose argument is proportional to πz. We thus find it convenient to introduce the primed quantities

$$R' = \frac{R}{\pi^4}, \quad T' = \frac{T}{\pi^4}, \quad Q' = \frac{Q}{\pi^2}, \quad \sigma = \frac{\sigma}{\pi^2}, \quad k' = \frac{k}{\pi}, \quad l' = \frac{l}{\pi}, \quad a' = \frac{a}{\pi},$$

and drop the primes from now onwards.

In table 1 we summarize the results for all the cases considered when

$$p \ll p_m \ll 1.$$

This particular situation is chosen since the modes preferred are overstable in most of the models. For other values of p and p_m the reader may consult the relevant section entered in the last column. In table 2, the different boundary layers present are given and their significance in adjusting the mainstream solutions is indicated by specifying the quantities they affect.

3. Model I

This model is defined by taking $\hat{\mathbf{B}}$ and $\hat{\mathbf{\Omega}}$ to be parallel to the z axis. The even solution of (2.2)–(2.5) is then

$$W = \sum_{j=1}^{6} A_j \cos \pi q_j z, \tag{3.1}$$

where the A_j $(j = 1, \ldots, 6)$ are constants and the q_j^2 $(j = 1, \ldots, 6)$ are the roots of

$$\left.\begin{aligned} (i\sigma p + h)\,[hS^2 + Tq^2(i\sigma p_m + h)^2] &= Ra^2(i\sigma p_m + h)\,S, \\ S = (i\sigma p_m + h)\,(i\sigma + h) + Qq^2, \quad h &= a^2 + q^2. \end{aligned}\right\} \tag{3.2}$$

Since the expression (3.1) describes the full solution of the system, the other variables θ, b, ξ and ζ are particular solutions of (2.2)–(2.5) (see Eltayeb 1972*b*, chap. 6). When we find these particular solutions and apply the boundary conditions, we obtain six linear homogeneous equations for the constants

$$A_j \quad (j = 1, \ldots, 6).$$

These equations possess a non-trivial solution for the A_j if and only if the determinant of their coefficients vanishes. When we set this determinant equal to zero, we obtain the characteristic equation for R and σ.

We shall now consider the different types of boundaries.

(*A*) *Free boundaries with finite conductivity*

Here the range of α, see (2.18), can be divided into the three intervals (i) $\alpha \leqslant \frac{5}{4}$, (ii) $\frac{5}{4} \leqslant \alpha \leqslant 3$ and (iii) $\alpha \geqslant 3$. We shall consider these cases individually.

Case (i). $\alpha \leqslant \frac{5}{4}$ or more precisely $T \leqslant T_1$, where

$$T_1^4 = 4{\cdot}14^7 p_m^6(1+p)^7(1+p_m)^3 Q^5/27\overline{p}^2(p_m+p)(p_m-p)^3(1-p_m)^4. \tag{3.3}$$

The critical mode here is the same as in the magnetic case treated by Gibson (1966), to leading order. We include the results here for comparison reasons:

$$\left.\begin{aligned} R_c = p^2(1+p_m)Q/p_m^2(1+p), \quad \sigma_c^2 = (p_m-p)Q/p_m^2(1+p), \quad p_m > p, \\ a_c^{24} = 4p^8p_m^2(p_m-p)Q^5/\overline{p}^2(1+p_m)^4(1+p)^5(p_m+p)^4. \end{aligned}\right\} \tag{3.4}$$

Case (ii, *a*). $\frac{5}{4} \leqslant \alpha \leqslant 2$. This case is defined by $T_1 \leqslant T \leqslant T_2$, where

$$T_2 = [162p^8(p_m-p)/E_c^7 p_m^5 \overline{p}^2(p+p_m)^2]^{\frac{1}{5}} Q^2, \tag{3.5}$$

T_1 is as before while E_c is defined by equation (3.14) below. The critical mode is obtained by assuming the orders of magnitude

$$R = O(Q), \quad \sigma^2 = O(Qa^2/T), \quad a^2 \ll \sigma \tag{3.6}$$

and using them to solve (3.2) for the q_j's. Once these have been found, we substitute them into the characteristic equation, again making use of the orders of magnitude (3.6), to find that, to leading order,

$$\left.\begin{aligned} R_c = \frac{2p^2Q}{p_m(p_m+p)}, \quad \sigma_c^2 = \frac{(p_m-p)Q^2a_c^2}{p_m^2(p_m+p)T}, \quad p_m > p, \\ a_c^7 = \frac{9p_m(p_m-p)^{\frac{1}{2}}}{8\overline{p}(p_m+p)^{\frac{1}{2}}}\left[\frac{(p_m-p)(p_m^2-1)Q}{p_m(p_m+p)^2T} + \frac{(p_m+p)^2T}{p^2Q^2}\right]^{-2}. \end{aligned}\right\} \tag{3.7}$$

The mainstream solution is then

$$W = \cos \pi z. \tag{3.8}$$

Before we proceed to the next case, we shall analyse the results obtained here and compare them with case (i) above. In case (i) the solution represents convective motions in a fluid layer composed of a mainstream solution characterized by the roots q_1 and q_2 of (3.2) and three boundary layers whose thicknesses are related to q_3, q_4 and q_5 (and q_6). The only difference between case (i) and the purely magnetic case (Gibson 1966) is that here the mainstream is governed by a fourth-order equation instead of a second-order equation and a double Hartmann layer replaces the single Hartmann layer.

In case (ii, *a*), the situation is different owing to the Coriolis forces being potent. It is interesting to examine the evolution of the boundary layers and the mainstream solution as the Coriolis forces increase. By applying the orders of magnitude (3.6) to (2.6), we find that the mainstream solution, taking $D = O(1)$, obeys the fourth-order differential equation

$$p(a^2Q^2D^2 + p_m^2\sigma^2T)D^2W = Rp_mQD^2W. \tag{3.9}$$

This equation represents a hydromagnetic gravity wave in a rotating inviscid fluid of zero thermal and magnetic diffusivities. This means that all forms of diffusion are absent from the main part of the layer.

Alternatively, the mainstream solution can be seen not to involve any form of diffusion by considering the thermal time scale of the critical mode. The thermal time scale τ_κ of the mainstream solution is of order L/κ, where $L = 2d/a_c$ is the width of a cell. The period of oscillation τ_0 is of order $2d^2/\nu\sigma_c$. Here a_c and σ_c are given by (3.7). It is readily seen that $\tau_\kappa \gg \tau_0$. The period of oscillation is thus too short for the temperature differences to diffuse out during a cycle. If the orders of magnitude of ν, κ and η are the same, the magnetic and viscous diffusion times are of the same order as the thermal diffusion time.

The structure of the boundary layer in case (ii, a) is similar to that in case (i) except that the double Hartmann layer is now a Hartmann–Ekman layer (see I).

Case (ii, b). $2 \leqslant \alpha \leqslant 3$. The analysis in this case applies for $T_2 \leqslant T \leqslant T_3$, where

$$T_3 = 27p^4Q^3/32(1+p)\,E_c^6, \quad E_c = 4{\cdot}2176. \tag{3.10}$$

Here the search for the orders of magnitude of R, σ and a for the critical mode leads to

$$R = O(T/Q), \quad a = O(1), \quad \sigma \leqslant O(1). \tag{3.11}$$

Employing these orders of magnitude in the characteristic equation, we find that the expression for R gives $R \to R_c$ as $\sigma \to 0$. When $\sigma = 0$, however, the system (2.2)–(2.7) factorizes into a tenth-order equation and a second-order equation for b. The solution is then that obtained for steady convection. Since this case of finite electrical conductivity was not discussed in I, we shall examine it here. Application of the relevant boundary conditions to the tenth-order system (neglecting the condition $\Lambda_1 = 0$) gives the characteristic equation

$$\tan(\tfrac{1}{2}q_2)/q_2^3 = -\tan(\tfrac{1}{2}q_3)/q_3^3, \tag{3.12}$$

where q_2 and q_3 are the roots of

$$(q^2+a^2)^2 = E^2a^2, \quad E^2 = RQ/T. \tag{3.13}$$

Equations (3.12) and (3.13) are solved numerically to obtain

$$E_c = 4{\cdot}217, \quad a_c = 2{\cdot}328. \tag{3.14}$$

The critical mode here is characterized by a sixth-order mainstream and *one* boundary layer. The mainstream solution is

$$W = 1 + 0{\cdot}6583\cos(6{\cdot}594z) + 0{\cdot}0012\cosh(12{\cdot}84z). \tag{3.15}$$

The boundary layer is the steady Ekman layer discussed in §3 of I.

The marginal state (3.14) has a wavenumber of the same order of magnitude as the thickness of the layer. The significance of this result is that the motions at marginal stability are three-dimensional. The Taylor–Proudman theorem and its analogue in hydromagnetics (see, for example, Hide & Roberts 1962) are no longer true. In the interior of the layer a balance is struck between the Lorentz, Coriolis and pressure forces while viscous dissipation is confined to thin layers on the boundaries. In contrast to the previous cases, the mainstream here is governed by an equation satisfied by a hydromagnetic gravity wave in a rotating, inviscid, electrically conducting fluid of non-vanishing thermal diffusivity. Such a wave is normally attenuated, a fact which means that σ is generally complex and not purely real.

We note that these results are in good agreement with the experimental work of Nagakawa (reported in Chandrasekhar 1961, chap. 5). By keeping the angular velocity constant ($T \sim 7 \times 10^5$) and increasing the strength of the magnetic field, he found a sudden enlargement of the cells at marginal stability when Q reached a certain value ($\sim 6 \times 10^2$).

When the boundaries are rigid (D), the above results for the critical modes hold good. The quantities adjusted by the boundary layers are shown in table 2.

(B) *Free insulating boundaries*

This type of boundary needs to be examined separately because the limit $\overline{p} \to 0$ is singular in the sense that, if we let $\overline{p} \to 0$ in the critical modes obtained for (A) above, we shall not get the critical modes for the case $\overline{p} = 0$.

Case (i). $\alpha \leqslant \frac{11}{10}$, or more precisely $T \leqslant T_1$, where

$$T_1^{10} = 4p^{12}(1+p_m)^{20}(p_m+2)^2 Q^{11}/(133)^4(1+p)^{30}(p_m-p)^{10}(p_m+p)p_m^5. \quad (3.16)$$

For this case application of the boundary conditions to the solution (3.1) above yields a critical mode in which R_c and σ_c are the same as in (A, i) above but the wavenumber is different:

$$a_c^5 = p^2(p_m+2)Q/2(1+p)(1+p_m)(p_m+p). \quad (3.17)$$

Case (ii, a). $\frac{11}{10} \leqslant \alpha \leqslant 2$. This case is found to consist of two subcases. If $T_1 \leqslant T \leqslant T_{12}$, where

$$T_{12}^2 = p^2(p_m-p)^2(1-p_m)^4 Q^3/p_m^2(1+p)^4(p_m+p)^5, \quad (3.18)$$

the critical mode possesses the R_c and σ_c given in (3.4) but has a different wavenumber:

$$a_c^{11} = 4p_m p^4(p_m-p)^2 T^2 Q/(1+p_m)^4(p_m+p)^2(1+p). \quad (3.19)$$

If, however, $T_{12} \leqslant T \leqslant T_2$, where

$$T_2 = \frac{p^2}{p_m^2}\left\{1+\left[1+\frac{p_m^2}{2(p_m+p)}\right]^{\frac{1}{2}}\right\}^2 Q^2, \quad (3.20)$$

then R_c and σ_c are identical to those in (3.7) but the critical wavenumber is given by

$$a_c = \left[\frac{(p_m+p)^2 T}{p^2 Q^2} + \frac{(p_m-p)(p_m^2-1)Q}{p_m(p_m+p)^2 T}\right]^{-\frac{1}{4}}. \quad (3.21)$$

The main features of these solutions resemble those of the corresponding case in (A) above. It may be noted, however, that the expression for T_{12} vanishes when $p_m = 1$. In this situation an alternative expression giving $T_{12} = O(Q^{\frac{5}{4}})$ is obtainable. The physical significance of this apparent singularity in the range of p_m, which we shall meet again later in model III, is not clear.

Case (ii, *b*). $2 \leqslant \alpha \leqslant 3$. When $T_2 \leqslant T \leqslant 27p^4Q^3/32(1+p)$, where T_2 is given by (3.20) above, the critical mode is steady and the results for the corresponding case in I apply.

(*C*) *Free, perfectly conducting boundaries*

The critical modes are qualitatively similar to the previous types but are included to allow easy comparison with models II and III, where only this type of boundary is examined.

Case (i). The critical mode is similar to that for the corresponding case with $\bar{p} = 0$ except that the expression on the right of (3.17) should be multiplied by a factor of 8.

Case (ii, *a*). The marginal state is again similar to that for insulating boundaries. The expression on the right of (3.19) is multiplied by a factor of 8 while a_c in (3.21) remains as it is.

Case (ii, *b*). The results here are identical to those for the corresponding case for boundaries of type *A* above.

To summarize the results for this model, the most unstable mode, according to the linear theory, is overstable if $p_m > p$ in both cases (i) and (ii, *a*). If $p_m \leqslant p$, however, the steady modes of I are preferred. In case (ii, *b*), the most unstable mode is always time independent whatever the relative magnitude of p and p_m. In case (iii) the results of I show that overstability is preferred if $p < (\frac{2}{3})^{\frac{1}{2}}$ and steady convection is preferred otherwise.

4. Model II

This model is defined by taking $\hat{\mathbf{\Omega}}$ parallel to the z axis and $\hat{\mathbf{B}}$ parallel to the x axis. In this model, we shall restrict the analysis to the case of perfectly conducting boundaries. For type *C* an exact solution exists. In the case *F* of rigid boundaries the same solution applies in the mainstream but boundary layers are present.

Consider boundaries of type *C*. Then the relevant boundary conditions can be shown to require that *W* and all its even derivatives (with respect to z) vanish. The *exact* (even) solution giving the smallest Rayleigh number is then

$$W = \cos \pi z. \tag{4.1}$$

If we substitute this solution in (2.5), properly adjusted for this model, and separate real and imaginary parts we get the following expression for R:

$$R = \frac{Z_1}{a^2}\left\{Z_1^2 - p\sigma^2 + \frac{Z_1^2 + pp_m\sigma^2}{Z_1^2+p_m^2\sigma^2}G + \frac{T}{Z_1}\frac{(Z_1^2 - p_m\sigma^2 + G)(Z_1^2 - pp_m\sigma^2) + (1+p_m)(p+p_m)Z_1^2\sigma^2}{(Z_1 - p_m\sigma^2+G)^2 + (1+p_m)^2\sigma^2 Z_1^2}\right\}, \tag{4.2}$$

where $G = Qk^2$, $Z_1 = 1+a^2$ and σ^2 is a root of the equation

$$\frac{T}{Z_1}\frac{(p-1)Z_1^2 - (p+p_m)G + (p-1)p_m^2\sigma^2}{(Z_1^2 - p_m\sigma^2 + G)^2 + (1+p_m)^2Z_1^2\sigma^2} + \frac{(p-p_m)G}{Z_1^2+p_m^2\sigma^2} + 1 + p = 0. \tag{4.3}$$

In this model, we find it convenient to divide the range of T and Q into the three intervals (i) $\alpha \leqslant 2$, (ii) $2 \leqslant \alpha \leqslant 3$ and (iii) $\alpha \geqslant 3$. Case (iii) will not be treated here since the results for the corresponding case in I apply.

Case (i). $\alpha \leqslant 2$, or more precisely $T \leqslant 4p^2Q^2/3(p_m+p)^2$. Here the critical mode is associated with the orders of magnitude

$$R = O(T^{\frac{1}{2}}), \quad \sigma = O(1), \quad a = O(1), \quad G^2 = O(T). \tag{4.4}$$

If we define the quantities R_1, T_1, τ_1 and q by the relations

$$R_1 = \tfrac{1}{2}R(q+1)\,q, \quad T_1 = T(q+1)^2, \quad \tau_1 = T_1/Q^2, \quad q = p_m/p, \tag{4.5}$$

it can be shown, without going into details, that in terms of these quantities the problem becomes identical to that of steady convection discussed in I (cf. equation (4.8) in I) plus an expression for σ. The results of I then apply, with proper adjustment, and the expression for σ_c is found to be

$$p_m^2\sigma_c^2 = q^2-2. \tag{4.6}$$

Equation (4.6) shows that overstability is possible only if $q^2 > 2$. If we compare the critical Rayleigh numbers for this case and that of steady convection, we find that, for all values of $q > 2$, this overstable mode is preferred to steady convection. For values of $q \leqslant \sqrt{2}$, however, steady convection is preferred since the only overstable mode is a roll parallel to $\mathbf{B}_0$ and this has a Rayleigh number ($= O(T^{\frac{2}{3}})$) which is too high. When $\sqrt{2} < q < 2$, both modes (steady and oscillatory) are present and have a critical Rayleigh number of order $T^{\frac{1}{2}}$. Since $\frac{1}{2}q(q+1)$ is greater than unity in this last case, it must be concluded that steady convection is the easier to excite.

Case (ii). $2 \leqslant \alpha \leqslant 3$. This mode is defined by $\tau_1 \geqslant \frac{4}{3}$ but $T \leqslant 54p^4Q^3/(1+p)$. The critical mode here is identical to that for steady convection dealt with in I (§4) if we use the quantities defined in (4.5) above, while σ_c^2 is given by

$$p_m^2\sigma_c^2 = (1+x_c)^2 x_c^{-1}(2q-2-x_c), \quad x_c = k_c^2. \tag{4.7}$$

Although it was shown that equations (4.13) and (4.14) of I yield a critical mode for the whole range of this case, the condition $\sigma_c^2 > 0$ imposes a constraint on these equations in the case of overstability. Since $x_c \sim \tau_1^{\frac{1}{3}}$ as $\tau \to \infty$, we see that for every value of q there exists a value τ_q of τ_1 such that overstable motions are non-existent if $\tau_1 > \tau_q$. In figure 1, we illustrate this result for some values of q. The curve labelled C is the steady mode discussed in I.

For each value of τ_1, it is found that, even when overstability is present, it has a critical Rayleigh number greater than that of the steady mode if q is less than a certain value $\bar{q}$, which depends on τ_1. We can therefore divide the τ_1, q plane into three distinct regions (see figure 2). In region I overstability is not possible. In region II overstable motions are present but not preferred, while in region III overstability is the easier to excite.

To locate the preferred mode here, we use figure 2. Starting with a prescribed q, we locate the value τ_q of τ_1 on the line dividing regions II and III which

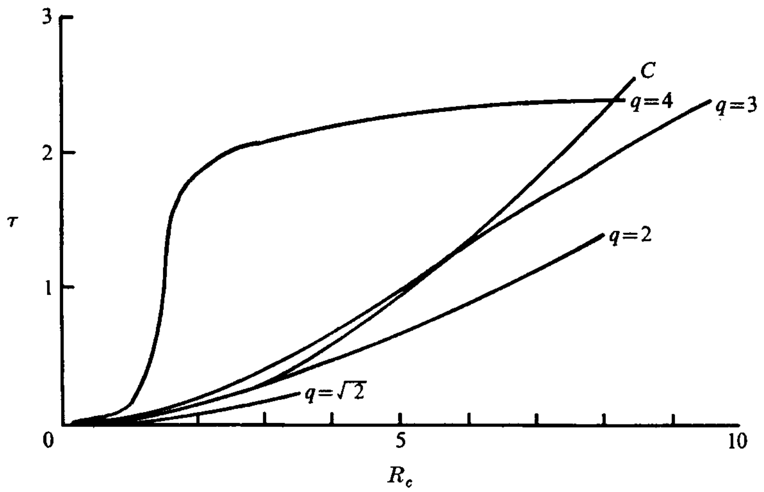

FIGURE 1. The variation of the critical Rayleigh number R_c with τ_1 ($= T_1/Q^2$) for different values of q ($= p_m/p$). For comparison, the steady-convection curve C is also included.

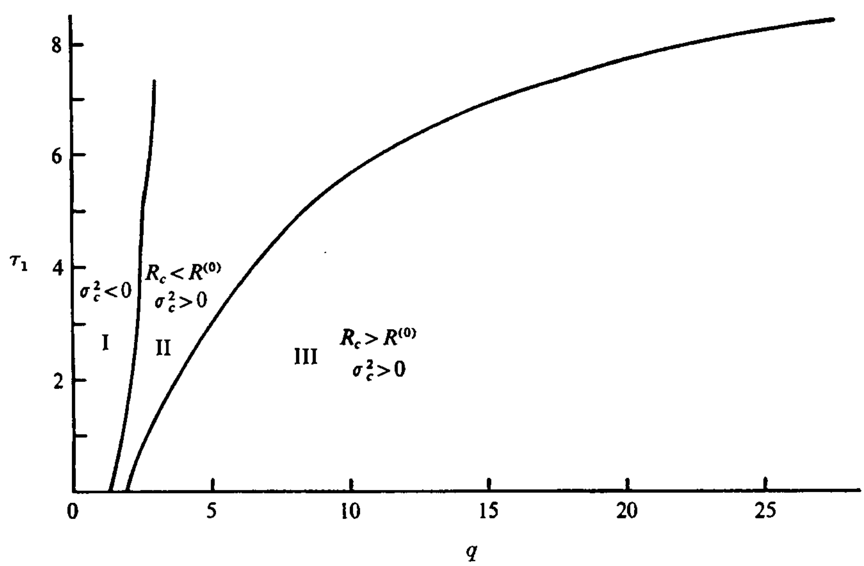

FIGURE 2. To illustrate the regions in the τ_1, q plane where overstability is preferred. In region I, overstable motions do not occur; in region II, overstability is present but steady convection is easier to excite, while overstability is preferred in region III. For each q, T_q is determined by $\tau_q = T_q/Q^2$, where $\tau_q = \tau$ when the curve for that value of q meets the curve C.

corresponds to q. Then, for this value of q, overstability is preferred if $\tau_1 < \tau_q$ and steady convection is preferred if $\tau_1 \geqslant \tau_q$.

In the case of rigid, perfectly conducting boundaries, the same critical modes are present. The solution (4.1) holds in the interior of the layer but an Ekman layer is present on the boundaries.

5. Model III

In this model, we take $\hat{\mathbf{B}}$ along the x axis and $\hat{\mathbf{\Omega}}$ in the x, y plane at an angle ϕ to $\hat{\mathbf{B}}$. The treatment here is for perfectly conducting boundaries only. Whether the boundaries are free or rigid, we find that the range of T can conveniently be divided into *two* intervals: (i) $\alpha \leqslant 2$ and (ii) $\alpha \geqslant 2$. The critical modes for rigid and free boundaries are found to be the same and hence the matching of cases (i) and (ii) is also the same. Here we shall deal with the mainstream solution only. If any information regarding the boundary layers is required, the reader may consult table 2.

Case (i). $\alpha \leqslant 2$. The precise matching values of this case with the next will be dealt with at the end of the discussion of case (ii).

If we use the quantities R_1, T_1 and τ_1 defined in (4.5) above, the problem becomes identical to that for the steady mode discussed in I if we replace R, T and τ by R_1, T_1 and τ_1 respectively. The expression for σ_c is then

$$p^2\sigma_c^2 = \frac{(1+x_c)^2(1-x_c)}{x_c}\left(q^2 - \frac{1}{1-x_c}\right), \quad x_c = a_c^2. \tag{5.1}$$

For small values of τ_1 (i.e. $\tau_1 \ll 1$), x_c takes the value $\frac{1}{2}$. Equation (5.1) then shows that overstability is present if $q^2 > 2$. A comparison of the steady and oscillatory modes shows that overstable motions are preferred for $q > 2$ while steady convection is preferred if $q \leqslant 2$, although overstability is present for $\sqrt{2} < q < 2$. This situation is akin to the corresponding case in model II.

As τ_1 increases (i.e. $\tau_1 = O(1)$), the expression for R_1 becomes

$$R_1 = h(x_c)\, Q, \tag{5.2}$$

where R_1 is defined in (4.5) above and $h(x_c)$ is a function of x_c which depends on the direction cosines of $\hat{\mathbf{\Omega}}$ and is given in equation (5.13) of I. The equation for x_c is also the same as in I. In figure 3, we illustrate the behaviour of $h(x_c)$ and τ_1 as functions of x_c in the relevant interval $0 < x < \frac{1}{2}$. We shall refer to this mode as the 'old oblique mode' since it is similar to the steady mode of I.

Case (ii). $\alpha \geqslant 2$. When $\tau_1 \gg 1$, the mode preferred depends on the relative magnitude of p and p_m. In fact it is always one of two modes.

The first mode is a roll parallel to the angular velocity vector $\mathbf{\Omega}$. We shall call this 'the magnetic roll'. Here the effect of the Coriolis force is absent. The basic equations then reduce to those for a Bénard layer in the presence of a horizontal magnetic field. The variables ξ and ζ vanish identically and the system of equations reduces to an eighth-order system. By applying the relevant boundary conditions, we find that an *exact* solution (given in (4.1) above) exists. The minimization procedure then leads to

$$\left.\begin{aligned} R_c = \frac{p^2(1+p)f^2Q}{p_m^2(1+p_m)}, \quad p_m^2\sigma_c^2 = \frac{(p_m-p)}{(1+p)}\left(\frac{f^2Q}{\alpha_1}\right)^{\frac{1}{2}},\\ x_c = (\alpha_1 Q)^{-\frac{1}{2}}, \quad k_c = -l_c\tan\phi, \end{aligned}\right\} \tag{5.3}$$

where $$\alpha_1 = p^2f^2/(1+p)(p+p_m), \quad f = \sin\phi. \tag{5.4}$$

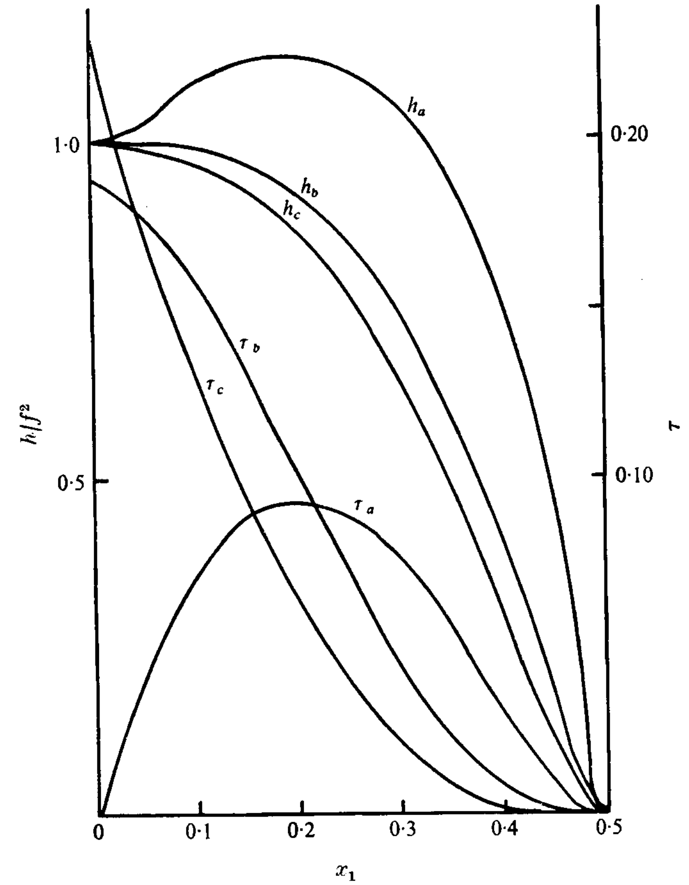

FIGURE 3. To illustrate the behaviour of the functions $h(x)$ and $\tau_1(x)$. The subscripts a, b and c indicate values for $g = 0$, 0·5 and 0·8 respectively.

The second mode is another oblique mode, referred to later as 'the modified oblique mode', and has no parallel in the steady-convection treatment of I. This mode is arrived at using the orders of magnitude

$$a = O(Q^{-\frac{1}{6}}), \quad R = O(Q), \quad \sigma = O(Q^{\frac{1}{6}}), \quad \sin(\psi+\phi) = O(Q^{-\frac{2}{3}}), \tag{5.5}$$

where $\tan\psi = k/l$. The equation for the mainstream is of second order and it is found to obey the boundary condition on θ, giving a solution identical to that in (4.1) above. The critical mode is then defined by

$$\left.\begin{aligned} R_c = 2p^2f^2Q/p_m(p_m+p), \quad a_c = \alpha_2(f^2Q)^{-\frac{1}{6}}, \\ \sin(\psi_c+\phi) = -\frac{\alpha_2^2}{4\tau_1}(f^2Q)^{-\frac{2}{3}}, \quad \sigma_c^2 = \frac{(p_m^2-p^2)}{p^2\alpha_2}(f^2Q)^{\frac{1}{3}}, \end{aligned}\right\} \tag{5.6}$$

where $\alpha_2 = (p_m-p)(p_m^2-p)/2p^2(p_m+p)$, provided that $p < \min(p_m, p_m^2)$.

We shall now discuss the matching of cases (i) and (ii). When $\tau_1 = O(1)$, the old oblique mode, when it exists, has a critical Rayleigh number which increases with τ_1. This can be deduced from the behaviour of $h(x_c)$ and τ_1 when plotted against x_c for different values of the angle ϕ (figure 3). When $\phi > 60°$, each of the two functions has two branches which meet at a certain value x_b (say) of x_c, depending on ϕ. This means that, for each value of $\tau_1 \leqslant \tau_1(x_b)$, there are two positive roots $\bar{x}$ and $\tilde{x}$ of x_c such that $0 < \bar{x} \leqslant x_b \leqslant \tilde{x} < \frac{1}{2}$, and each of these values corresponds to a value of R_{1c}. Also, the root $\bar{x}$ yields an overstable mode only if $q^2 > \bar{q} = (1-\bar{x})^{-1}$ while the value $\tilde{x}$ gives an overstable mode only if $q^2 > \tilde{q} = (1-\tilde{x})^{-1}$. Since $\tilde{x} > \bar{x}$, we clearly see that $\tilde{q} > \bar{q}$. This shows that for $\tilde{q} < q^2 \leqslant \bar{q}$ there is only one overstable mode and it corresponds to $\bar{x}$. When $q^2 > \tilde{q}$, however, the two oscillatory modes corresponding to $\tilde{x}$ and $\bar{x}$ occur. A closer examination of these modes shows that the root $\tilde{x}$ corresponds to the smaller critical Rayleigh number. If $\phi \leqslant 60°$, h and τ_1 decrease steadily, as x_c is increased from zero, until they approach zero when x_c tends to $\frac{1}{2}$ from below. Hence, for each τ_1 less than a certain value τ_s (say), there exists *one* root x' which gives a critical oscillatory mode provided that $q^2 > 1/(1-x')$. We therefore conclude that the old oblique mode is possible only if $\tau_1 \leqslant \tau_1(x_b)$ if $\phi > 60°$ and $\tau_1 \leqslant \tau_s$ if $\phi \leqslant 60°$, where $\tau_1(x_b)$ and τ_s are the maximum points on the corresponding τ_1 curves.

To determine the critical mode globally, we find that there are three cases to consider: (*a*) $p_m \leqslant p$, (*b*) $p < p_m < 2p$ and (*c*) $p_m \geqslant 2p$.

(*a*) In this situation, the most unstable modes are the steady convection modes discussed in I.

(*b*) When $p < p_m < 2p$, the critical mode is the steady mode of I for all $\tau_1 \leqslant \tau_\phi$. When $\tau_1 = \tau_\phi$, the steady oblique mode matches with the modified oblique mode if $p_m \geqslant 1$ and with the magnetic roll if $p_m < 1$. A sample of matching quantities is given in tables 3 and 4.

(*c*) If $p_m \geqslant 2p$ (i.e. $q < 2$), the preferred mode is overstable for all values of τ_1. For small values of τ_1, the old oblique mode of case (i) above is preferred. This mode matches with the modified oblique mode at $\tau_1 = \tau_\phi$ if $p_m \geqslant 1$. The values of τ_ϕ are the same as those given in table 3 in I if $\phi > 60°$ while $\tau_\phi = f^2(1-f^2)$ if $\phi \leqslant 60°$. If $p_m < 1$, the old oblique mode matches with the magnetic roll.

6. Concluding remarks

The study of the four models defined in this paper has shown some characteristics of the coupling between the Coriolis and Lorentz forces. Consider a rotating Bénard layer. The critical mode has a large wavenumber ($a_c \sim T^{\frac{1}{6}}$), indicating that the motions are two-dimensional. Also the critical Rayleigh number R_c, which is the measure of the temperature gradient β_c required for instability, varies as $T^{\frac{2}{3}}$. This means that $\beta_c \sim \bar{\nu}^{\frac{1}{3}}$. If a magnetic field is introduced into the layer, the critical mode remains the same to leading order until the magnetic field reaches a certain value $[T = O(Q^3)]$, when the wavenumber starts to decrease. When $T = O(Q^2)$, the critical Rayleigh number R_c is *drastically reduced* to $O(T^{\frac{1}{2}})$ and the wavenumber a becomes $O(1)$. The latter shows that the motions at marginal stability are three-dimensional while the former indicates that the critical temperature gradient β_c necessary for instability is greatly reduced and also that β_c is independent of viscosity.

$g\ (= \cos\phi)$	$q\ (= p_m/p)$	τ_ϕ	x_ϕ	$\lvert\psi^0_\phi\rvert$
0·0	2·00	0·0047	0·468	20·1
	1·50	0·0130	0·443	26·8
	1·09	0·0437	0·375	39·2
0·2	2·00	0·0051	0·448	20·5
	1·50	0·0158	0·412	27·5
	1·09	0·0581	0·315	40·5
0·8	2·00	0·0031	0·395	13·1
	1·50	0·0111	0·340	17·9
	1·09	0·0666	0·195	27·5

TABLE 3. The values of τ_1, x_c and ψ when steady convection matches with the new oblique mode in the case $p_m > 1$ and $p < p_m < 2p$

$g\ (= \cos\phi)$	p_m	p	τ_ϕ	x_ϕ	$\lvert\psi^0_\phi\rvert$
0·0	0·8	0·5	0·0103	0·450	25·2
	0·6	0·31	0·0041	0·470	19·7
	0·6	0·5	0·0294	0·405	34·4
	0·4	0·3	0·0181	0·430	29·7
0·2	0·8	0·5	0·0101	0·425	25·3
	0·6	0·31	0·0047	0·450	19·9
	0·6	0·5	0·0375	0·360	35·5
	0·4	0·3	0·0202	0·400	29·1
0·8	0·8	0·5	0·0074	0·360	16·3
	0·6	0·31	0·0026	0·400	12·7
	0·6	0·5	0·0301	0·265	23·3
	0·4	0·3	0·0156	0·320	19·4

TABLE 4. The values of τ_1, x_c and ψ when the steady mode matches with the magnetic roll in the case $p_m < 1$ and $p < p_m < 2p$

Although the results obtained for the different models are qualitatively similar, one can see disagreement when comparing the results quantitatively. For example, when $T \ll Q^2$, the results of models I and IV differ from these of models II and III in that a balance is maintained between Lorentz and Coriolis forces in models II and III while the Lorentz forces dominate the Coriolis forces in the other two models. Furthermore, the critical mode of model III possesses non-zero helicity, which is known to be conducive to hydromagnetic dynamo action (Moffatt 1970), whereas that of model I has zero helicity.

It must be noted, however, that in more realistic models of physical significance (i.e. the earth), the situation is more complicated. In general, the magnetic field and gravity are not only inclined to the axis of rotation but are also non-uniform. Nevertheless, we may expect these results to be qualitatively applicable, realizing that model II may be representative of the polar regions while model III may be representative of an equatorial belt (Eltayeb & Kumar 1975).

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